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Solving First-Order Differential Equations Analytically and Numerically

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Abstract: First-order differential equations are fundamental tools in modeling physical, biological, and engineering systems. This paper explores both analytical and numerical approaches to solving first-order differential equations. Analytical methods such as separation of variables, integrating factors, and exact equations provide exact solutions under ideal conditions, while numerical techniques like Euler's method and Runge-Kutta methods allow approximate solutions when exact forms are difficult or impossible to find. Examples are presented to illustrate the application of each method.

Introduction

Differential equations are essential in describing dynamic systems in various scientific fields. A first-order differential equation involves the first derivative of a function and provides a relationship between a function and its rate of change. Solving these equations is crucial for understanding system behaviors over time. While analytical solutions offer closed-form expressions, numerical methods are often used when equations are too complex for exact solutions.

Common Methods for Solving First-Order Differential Equations

1. Separation of Variables

This method is used when the equation can be expressed as a product of a function of x and a function of:

$$dy/dx = g(x) * h(y)$$

Rewriting gives:

$$(1/h(y))\, dy = g(x)\, dx$$

Then integrate both sides to find the solution.

Example 1
Given: $dy/dx = x + y$
This equation is not directly separable. To solve it, we rearrange it using substitution.
Let $v = y * e^{(-x)}$, $then\ y = v * e^{(x)}$
Then $dy/dx = dv/dx * e^{(x)} + v * e^{(x)}$
Substitute into the original equation:
$dv/dx * e^{(x)} + v * e^{(x)} = x + v * e^{(x)}$
Cancel $v * e^{(x)}$ from both sides:
$dv/dx * e^{(x)} = x$
$dv/dx = x * e^{(-x)}$
Integrate both sides:
$v = \int x * e^{(-x)}\, dx$ (Use integration by parts)
$v = -x * e^{(-x)} - e^{(-x)} + C$
Then $y = v * e^{(x)} = (-x - 1 + C * e^{(x)})$
Apply the initial condition $y(0) = 1$ to find C.

Example 2
Given: $dy/dx = y - x^2 + 1$
This equation is not directly separable, but can be solved as a linear first-order equation.
Rewriting: $dy/dx - y = -x^2 + 1$
This is a linear **ODE**. The integrating factor is $\mu(x) = e^{(-x)}$
Multiply both sides:
$e^{(-x)}\, dy/dx - y * e^{(-x)} = (-x^2 + 1) * e^{(-x)}$

$d/dx\,(y * e^{(-x)}) = (-x^2 + 1) * e^{(-x)}$
Integrate both sides to find y.

Example 3
Given: $dy/dx = sin(x) + y$
This is also a linear first-order equation.
Rewriting: $dy/dx - y = sin(x)$
The integrating factor is $\mu(x) = e^{(-x)}$
Multiply both sides:
$e^{(-x)}\,dy/dx - y * e^{(-x)} = sin(x) * e^{(-x)}$
$d/dx\,(y * e^{(-x)}) = sin(x) * e^{(-x)}$
Integrate both sides to solve for y.

2. Homogeneous Equations
An equation of the form $dy/dx = F(y/x)$ is called homogeneous.
Use the substitution:

Solving Differential Equations Using the Method of Homogeneous Equations

Example 1:
Given: $dy/dx = x + y, y(0) = 1$
This equation is not homogeneous. To convert it into a homogeneous form, we perform the substitution:
Let $v = y / x \Rightarrow y = xv$
Then $dy/dx = x\,dv/dx + v$
Substitute into the equation:
$x\,dv/dx + v = x + xv$
$x\,dv/dx + v = x(1 + v)$
$x\,dv/dx = x(1 + v) - v = x + xv - v = x + v(x - 1)$
This is still not separable, so we conclude that this equation is better approached with other methods (e.g., numerical methods or integrating factor).

Example 2:
Given: $\frac{dy}{dx} = y - x^2 + 1, y(0) = 0.5$,
This equation is also not homogeneous. It is a first-order linear differential equation.
Rewriting:
$dy/dx - y = -x^2 + 1$
Use the integrating factor method:
Integrating factor $(I.F.) = e^{-x}$
Multiply both sides by **I.F.:**
$e^{-x}\,dy/dx - e^{-x}y = (-x^2 + 1)e^{-x}$
$\rightarrow d/dx\,[e^{-x}\,y] = (-x^2 + 1)e^{-x}$
Integrate both sides to find y.

Example 3:
Given: $\frac{dy}{dx} = sin(x) + y, y(0) = 1$
Again, not a homogeneous equation. It is a linear differential equation.
Rewrite:
$$\frac{dy}{dx} - y = sin(x)$$
Integrating factor $= e^{-x}$
Multiply both sides:
$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\,sin(x)$$
$$\rightarrow \frac{d}{dx}[e^{-x}y] = e^{-x}\,sin(x)$$
Integrate both sides to find y.

Example 4:

Solve the differential equation:

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

Solution: This is a homogeneous differential equation because the right-hand side is a function of $(\frac{y}{x})$.

Let $v = \frac{y}{x}$, *hence* $y = vx$.

Differentiate both sides with respect to x: $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute into the original equation:

$$v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$$

Rewriting: $x\frac{dv}{dx} = \frac{1+v}{1-v} - v$

Combine the terms:

$$x\frac{dv}{dx} = \frac{1 + v - v(1 - v)}{1 - v} = \frac{1 + v^2}{1 - v}$$

Separate variables: $\frac{1-v}{1+v^2}dv = \frac{1}{x}dx$

Integrate both sides:

Left side:

$$\int \frac{1 - v}{1 + v^2}dv = \int \frac{1}{1 + v^2}dv - \int \frac{v}{1 + v^2}dv = arctan(v) - 0.5\, ln(1 + v^2)$$

Right side: $\int \frac{1}{x}dx = ln|x|$

Combine the results: $arctan(v) - 0.5\, ln(1 + v^2) = ln|x| + C$

Replace $v = \frac{y}{x}$: $arctan\left(\frac{y}{x}\right) - 0.5\, ln\left(1 + \left(\frac{y}{x}\right)^2\right) = ln|x| + C$

Example 5:

Solve the differential equation:

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$$

Solution: This is also a homogeneous differential equation.

Let $v = \frac{y}{x}$, *then* $y = vx$, *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute: $v + x\frac{dv}{dx} = v + v^2$

Subtract v from both sides: $x\frac{dv}{dx} = v^2$

Separate variables: $\frac{1}{v^2}dv = \frac{1}{x}dx$

Integrate both sides: $\int v^{-2}dv = \int x^{-1}dx \rightarrow -v^{-1} = ln|x| + C$

Substitute back $v = \frac{y}{x}$: $-\frac{x}{y} = ln|x| + C$

Or equivalently: $\frac{x}{y} = -ln|x| - C$

3. Solving Differential Equations Using the Integrating Factor Method

Example 1:

Given: $dy/dx = x + y, y(0) = 1$

Step 1: Rewrite the equation in standard linear form:

$dy/dx - y = x$

Step 2: Find the integrating factor ($I.F.$):

$I.F. = e^{-\int 1dx} = e^{-x}$

Step 3: Multiply both sides by the $I.F.$:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = x\,e^{-x}$$

$$\rightarrow \frac{d}{dx}[e^{-x}y] = x\,e^{-x}$$

Step 4: Integrate both sides:

$$\int \frac{d}{dx}[e^{-x}y]dx = \int x\,e^{-x}dx$$

Use integration by parts:
$\int x e^{-x} dx = -x e^{-x} - e^{-x} + C$

So, $e^{-x} y = -x e^{-x} - e^{-x} + C$
Step 5: Solve for y :
$y = -x - 1 + C e^{x}$
Step 6: Apply initial condition $y(0) = 1$:
$1 = -0 - 1 + C \rightarrow C = 2$

Final solution: $y = -x - 1 + 2e^{x}$

Example 2:
Given: $\frac{dy}{dx} = y - x^2 + 1, y(0) = 0.5$
Step 1: Rewrite the equation in standard linear form:
$\frac{dy}{dx} - y = -x^2 + 1$
Step 2: Find the integrating factor ($I.F.$):
$I.F. = e^{-x}$
Step 3: Multiply both sides by the $I.F.$:
$e^{-x}\frac{dy}{dx} - e^{-x} y = (-x^2 + 1)e^{-x}$
$\rightarrow \frac{d}{dx}[e^{-x} y] = (-x^2 + 1)e^{-x}$
Step 4: Integrate both sides:

$$\frac{\int 1\, d}{dx}[e^{-x}y]dx = \int 1\,(-x^2 + 1)e^{-x}\, dx$$

Split the integral:
$\int 1\,(-x^2 + 1)e^{-x}\, dx = -\int 1\, x^2 e^{-x}\, dx + \int 1\, e^{-x}\, dx$
Use integration by parts (twice) or a symbolic tool:
$\int 1\, x^2 e^{-x}\, dx = (x^2 + 2x + 2)\, e\hat{\ }(-x)$
So the full integral = $[-(x^2 + 2x + 2) + 1]e^{-x} + C$
Then, $e^{-x} y = [-(x^2 + 2x + 2) + 1]e^{-x} + C$
$\rightarrow y = -x^2 - 2x - 1 + C e^{x}$

Step 5: Apply initial condition $y(0) = 0.5$:
$0.5 = -0 - 0 - 1 + C \rightarrow C = 1.5$
Final solution: $y = -x^2 - 2x - 1 + 1.5e^{x}$

Example 3:
Given: $dy/dx = sin(x) + y, y(0) = 1$

Step 1: Rewrite the equation in standard linear form:
$dy/dx - y = sin(x)$

Step 2: Find the integrating factor ($I.F.$):
$I.F. = e^{-x}$

Step 3: Multiply both sides by the $I.F.$:
$e^{-x}\, dy/dx - e^{-x} y = sin(x)e^{-x}$
$\rightarrow d/dx\,[e^{-x} y] = sin(x)e^{-x}$

Step 4: Integrate both sides:
$\frac{\int 1\, d}{dx}[e^{-x} y]dx = \int sin(x)\, e^{-x}\, dx$

Use integration by parts or known result:

$$\int \boldsymbol{sin(x)\, e^{-x}\, dx = -0.5\, e^{-x}\,(sin(x) + cos(x)) + C}$$

So, $\boldsymbol{e^{-x}\, y = -0.5\, e^{-x}\,(sin(x) + cos(x)) + C}$
$\boldsymbol{\rightarrow y = -0.5\,(sin(x) + cos(x)) + C\, e^{x}}$

Step 5: Apply initial condition **y(0) = 1:**
$\boldsymbol{1 = -0.5\,(0 + 1) + C \rightarrow C = 1.5}$
Final solution: $\boldsymbol{y = -0.5\,(sin(x) + cos(x)) + 1.5\, e^{x}}$
There are other methods for solving first-order differential equations, including:

4. Linear Equations
Form: $\boldsymbol{\frac{dy}{dx} + P(x)y = Q(x)}$
Solution: Use the integrating factor $\boldsymbol{\mu(x) = e^{\int P(x)dx}}$
Multiply through and solve:

$$\boldsymbol{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \rightarrow \frac{d}{dx[\mu(x)y\,]} = \mu(x)Q(x)}$$

5. Exact Equations
Form: $\boldsymbol{M(x,y)dx + N(x,y)dy = 0}$
Condition: $\boldsymbol{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$
Solution: Find a function $\boldsymbol{\psi(x,y)}$ such that:

$$\boldsymbol{\frac{\partial \psi}{\partial x} = M, \frac{\partial \psi}{\partial y} = N}$$

Then $\boldsymbol{\psi(x,y) = C}$

6. Bernoulli Equation
Form: $\boldsymbol{\frac{dy}{dx} + P(x)y = Q(x)y^{n}}$
Solution: Make substitution $\boldsymbol{v = y^{1-n}}$, then reduce to a linear equation

Numerical Methods for Solving First-Order Differential Equations

First-order differential equations are fundamental in modeling various natural phenomena. Often, these equations cannot be solved analytically, and numerical methods provide approximate solutions. In this document, we will explore three numerical methods: Euler's Method, Improved Euler's Method (Heun's Method), and the Runge-Kutta Method (RK4). Each method will be applied to the same three example problems to allow for comparison.

First-order differential equations involve derivatives of the first order only. The general form is:

$\boldsymbol{dy/dx = f(x,y),}$
with an initial condition

$$\boldsymbol{y(x_0) = y_0.}$$

Example Problems Used in All Methods:
Example 1: $\mathbf{\frac{dy}{dx} = x + y\ ,\ y(0) = 1\ ,\ h = 0.1}$
Example 2: $\mathbf{\frac{dy}{dx} = y - x^2 + 1\ ,\ y(0) = 0.5\ ,\ h = 0.2}$
Example 3: $\mathbf{\frac{dy}{dx} = sin(x) + y\ ,\ y(0) = 1\ ,\ h = 0.1}$

Some analytical methods for solving equations:

1. Euler's Method

Euler's method is the simplest numerical method. It uses the slope at the current point to estimate the next point.

Formula:

$y_{n+1} = y_n + h * f(x_n, y_n)$

Where h is the step size.

Pros: Simple and easy to implement.

Cons: Not very accurate unless the step size is very small.

Example 1: $\frac{dy}{dx} = x + y$

We will solve this using Euler's Method.

Euler's formula is:

$y_{n+1} = y_n + h * f(x_n, y_n)$

Where $f(x, y) = x + y.$

Let the initial conditions be:

$x_0 = 0 , y_0 = 1$

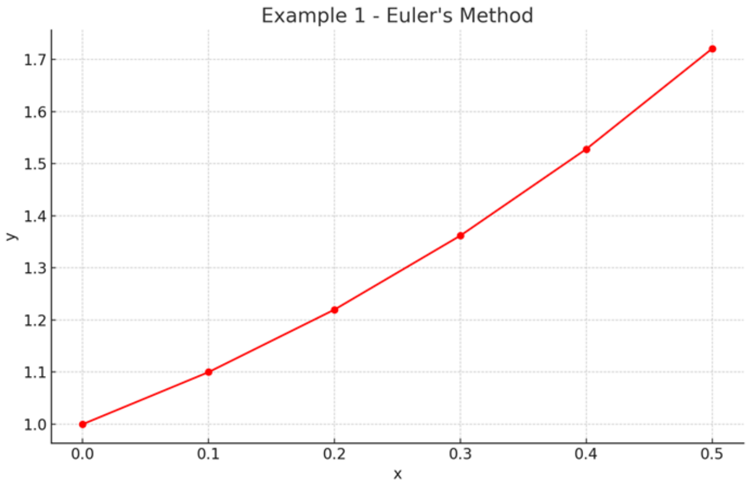
Step size $h = 0.1$

We will compute y for 5 steps.

Step-by-step Calculations :

Step (n)	x_n	y_n	$f(x_n, y_n) = x_n + y_n$
0	**0.00**	**1.0000**	**1.0000**
1	**0.10**	**1.1000**	**1.2000**
2	**0.20**	**1.2200**	**1.4200**
3	**0.30**	**1.3620**	**1.6620**
4	**0.40**	**1.5282**	**1.9282**

After **5** steps, the approximate value of y at $x = 0.5$ is $1.7210.$



Example 2: $\frac{dy}{dx} = y - x^2 + 1$

$dy/dx = y - x^2 + 1$

Initial condition $x_0 = 0 , y_0 = 0.5$

Step size: $h = 0.2$

Using Euler's method:

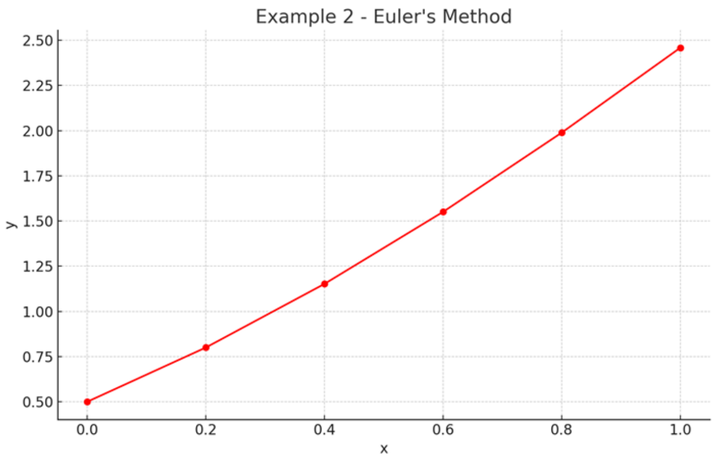
$$y_{n+1} = y_n + h * f(x_n, y_n)$$

Where

$f(x, y) = y - x^2 + 1$

Step (n)	x_n	y_n	$f(x_n, y_n)$
0	**0.00**	**0.5000**	**1.5000**
1	**0.20**	**0.8000**	**1.7600**
2	**0.40**	**1.1520**	**1.9920**
3	**0.60**	**1.5504**	**2.1904**
4	**0.80**	**1.9885**	**2.3485**

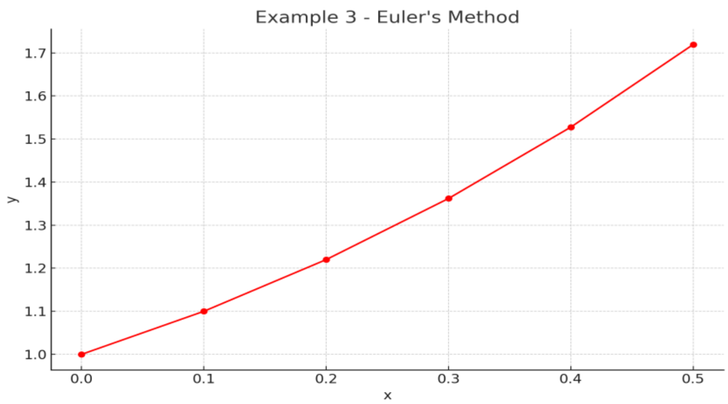
Approximate value after 5 steps: $y(1.0) \approx 2.4582$



Example 3: $dy/dx = sin(x) + y$
Initial condition $x_0 = 0$, $y_0 = 1$
Step size: $h = 0.1$, Using Euler's method:
$y_{n+1} = y_n + h * f(x_n, y_n)$
Where $f(x, y) = sin(x) + y$

Step (n)	x_n	y_n	$f(x_n, y_n)$
0	**0.00**	**1.0000**	**1.0000**
1	**0.10**	**1.1000**	**1.1998**
2	**0.20**	**1.2200**	**1.4187**
3	**0.30**	**1.3618**	**1.6574**
4	**0.40**	**1.5276**	**1.9170**

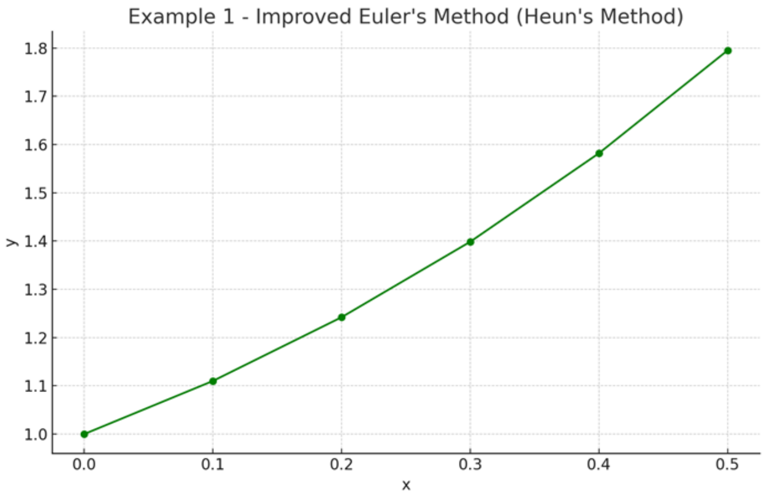
Approximate value after **5** steps: $y(0.5) \approx 1.7193$



2. Improved Euler's Method (Heun's Method)
Heun's method improves upon Euler's method by averaging slopes.
Formula:
Predictor: $y* = y_n + h * f(x_n, y_n)$
Corrector: $y_{n+1} = y_n + (h/2) * [f(x_n, y_n) + f(x_{n+1}, y*)]$
This approach reduces the error significantly compared to Euler's method.

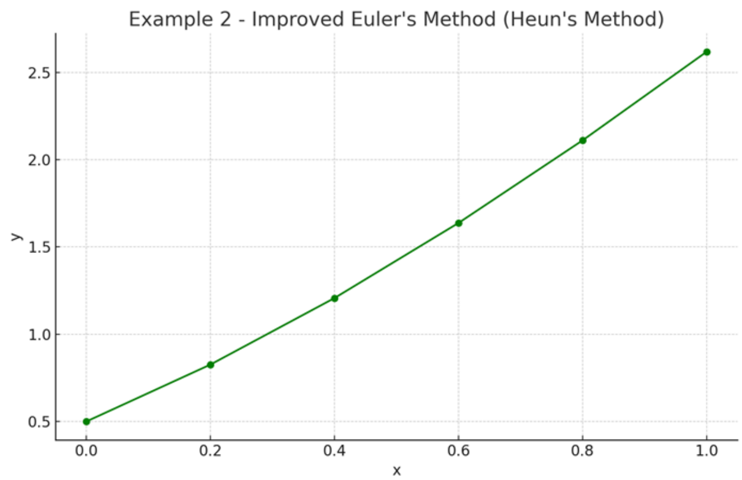
Example 1
Differential Equation: $dy/dx = x + y$
Initial Condition: $y(0) = 1$
Step size: $h = 0.1$

x_n	x_n	Predictor y*	$f(x_n, y_n)$	$f(x_{n+1}, y*)$
0.00	**1.00000**	**1.10000**	**1.00000**	**1.20000**
0.10	**1.11000**	**1.23100**	**1.21000**	**1.43100**
0.20	**1.24205**	**1.38626**	**1.44205**	**1.68626**
0.30	**1.39847**	**1.56831**	**1.69847**	**1.96831**
0.40	**1.58180**	**1.77998**	**1.98180**	**2.27998**



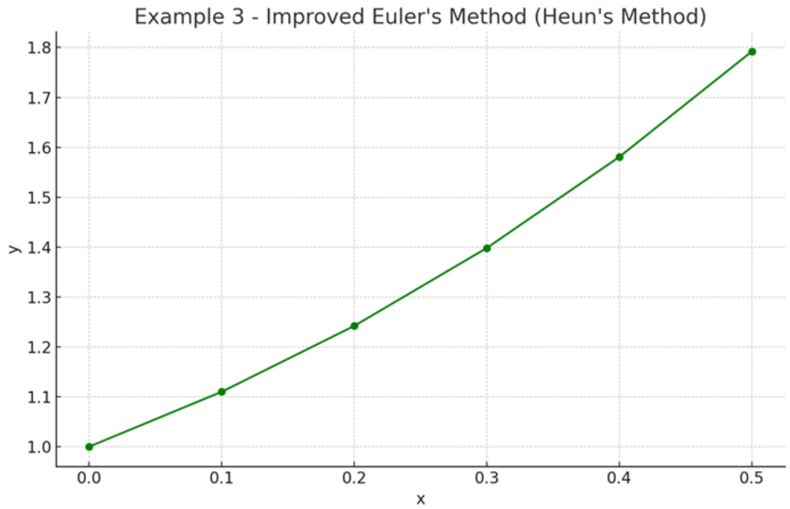
Example 2
Differential Equation: $dy/dx = y - x^2 + 1$
Initial Condition: $y(0) = 0.5$
Step size: $h = 0.2$

x_n	y_n	Predictor y*	$f(x_n, y_n)$	$f(x_{n+1}, y*)$
0.00	**0.50000**	**0.80000**	**1.50000**	**1.76000**
0.20	**0.82600**	**1.18320**	**1.78600**	**2.02320**
0.40	**1.20692**	**1.61630**	**2.04692**	**2.25630**
0.60	**1.63724**	**2.09269**	**2.27724**	**2.45269**
0.80	**2.11024**	**2.60428**	**2.47024**	**2.60428**



Example 3
Differential Equation: $dy/dx = sin(x) + y$
Initial Condition: $y(0) = 1$
Step size: ***h = 0.1***

x_n	y_n	Predictor y*	$f(x_n, y_n)$	$f(x_{n+1}, y*)$
0.00	1.00000	1.10000	1.00000	1.19983
0.10	1.10999	1.23097	1.20983	1.42964
0.20	1.24197	1.38603	1.44063	1.68155
0.30	1.39807	1.56743	1.69359	1.95685
0.40	1.58060	1.77760	1.97001	2.25702



3. Runge-Kutta Methods

The Runge-Kutta methods are a family of higher-order methods. The most commonly used is the **4th-order** method ($RK4$).

Formula:

$k_1 = h * f(x_n, y_n)$

$k_2 = h * f(x_n + h/2, y_n + k_1/2)$

$k_3 = h * f(x_n + h/2, y_n + k_2/2)$

$k_4 = h * f(x_n + h, y_n + k_3)$

$y_{n+1} = y_n + (1/6) * (k_1 + 2k_2 + 2k_3 + k_4)$

$RK4$ is accurate and widely used in engineering and scientific computing.

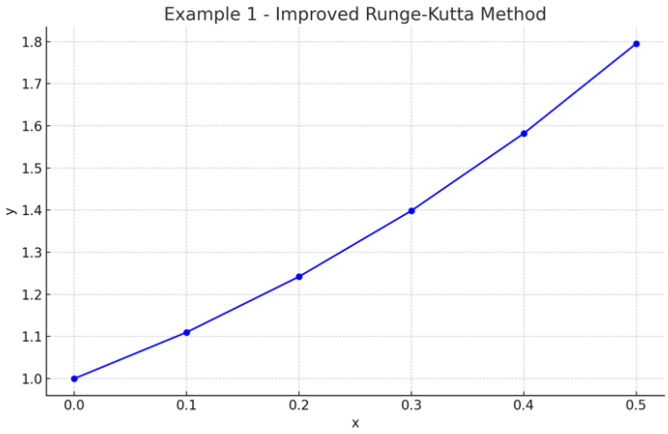
Example 1

Differential Equation: $dy/dx = x + y$

Initial Condition: $y(0) = 1$,

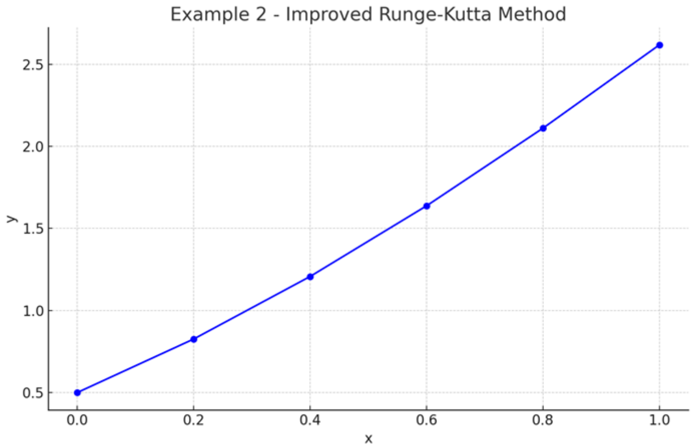
Step size: $h = 0.1$

Step	*x*	*y*	*k1*	*k2*	*y_new*
1	0	1	1	1.2	1.11
2	0.1	1.11	1.21	1.431	1.2421
3	0.2	1.2421	1.4421	1.6863	1.3985
4	0.3	1.3985	1.6985	1.9683	1.5818
5	0.4	1.5818	1.9818	2.28	1.7949



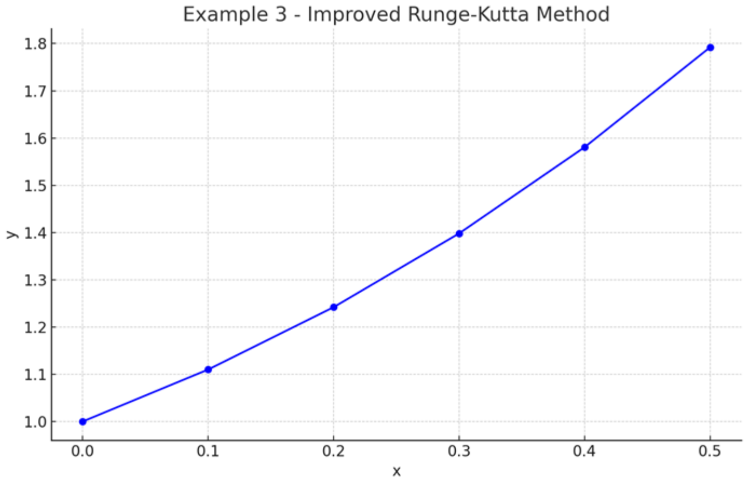
Example 2
Differential Equation: $dy/dx = y - x^2 + 1$
Initial Condition: $y(0) = 0.5$
Step size: $h = 0.2$

Step	***x***	***y***	***k1***	***k2***	***y_new***
1	**0**	**0.5**	**1.5**	**1.76**	**0.826**
2	**0.2**	**0.826**	**1.786**	**2.0232**	**1.2069**
3	**0.4**	**1.2069**	**2.0469**	**2.2563**	**1.6372**
4	**0.6**	**1.6372**	**2.2772**	**2.4527**	**2.1102**
5	**0.8**	**2.1102**	**2.4702**	**2.6043**	**2.6177**



Example 3
Differential Equation: $dy/dx = sin(x) + y$
Initial Condition: $y(0) = 1$
Step size: ***h = 0.1***

Step	***x***	***y***	***k1***	***k2***	***y_new***
1	**0**	**1**	**1.0**	**1.1998**	**1.11**
2	**0.1**	**1.11**	**1.2098**	**1.4296**	**1.242**
3	**0.2**	**1.242**	**1.4406**	**1.6815**	**1.3981**
4	**0.3**	**1.3981**	**1.6936**	**1.9569**	**1.5806**
5	**0.4**	**1.5806**	**1.97**	**2.257**	**1.7919**



Analysis of Results

we analyze the results of three different numerical methods used to solve first-order ordinary differential equations (ODEs): Euler's Method, Improved Euler's Method (Heun's Method), and the Improved Runge-Kutta Method (2nd Order). Each method was applied to three example problems with initial conditions and specific step sizes. We compare the accuracy and reliability of the results obtained.

Euler's Method is a first-order numerical procedure for solving ODEs. It is simple but less accurate, especially with larger step sizes.

This method improves upon Euler's Method by averaging slopes. It is a second-order method and provides better accuracy.

The Improved Runge-Kutta method (2nd order) provides even better accuracy by using a weighted average of slopes.

The results from the three methods for each example were tabulated. Euler's method generally gave the least accurate results. Improved Euler's method showed a significant improvement, while the Improved Runge-Kutta method provided the most accurate estimates, with smaller error margins due to the use of intermediate slopes.

Conclusion

In conclusion, the Improved Runge-Kutta method proved to be the most reliable and accurate among the three. For practical applications where precision is important, this method is recommended. Euler's method can be used for quick, rough estimates, but it is not suitable for problems where high accuracy is needed.

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