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Towards Reducing the Difficulties of Students in Fractions: Instructional Practices and Theoretical Context

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Abstract: In spite of the fact that curriculum change and mathematical school texts adapt to new education needs, students, internationally, continue to have difficulties when handling fractions. The present paper aims to present education practices which within a period of five years were applied by the research team. Our proposals help to reduce the above student difficulties. These teaching practices take into account the results as stated by international bibliographies as well as years of research of our team on rational numbers. They emphasize on multiple representations, use of experiential activities and activities carried out on electronic platforms. Additionally, our proposals are based on methods to reduce the causes of these difficulties. Thus, we question prospective teachers' beliefs on rational numbers, the structure and the context of Greek school textbooks, the curriculum and the instructional practices and approaches of various international researchers.
Keywords: difficulties, fractions, instructional practices, multiple representations

1. Introduction

What are the difficulties students faces when it comes to fractions? What are the possible causes of these difficulties? Can these difficulties be addressed and how? This paper attempts to answer the above questions focusing on the improper fractions, on the notions of fraction sequences as representations on the number line and on the division of the unit fraction into equal parts. These concepts were chosen because several researchers think that they are essential for developing rational number meaning and also they are associated with the understanding of other mathematical meanings [22]. In particular, Lee and Shin [26] indicate that the distributive partitioning operation was revealed in various mathematical problem situations such as fraction multiplication, fraction division, and multiplicative transformation between fractional quantities. Additionally, the knowledge of improper fractions associates with problem posing [3]. Moreover, implicit use of fractional can lead to more explicit use of structures and relationships in algebraic situations [13, 17]. The fractional knowledge, that is, influenced how students wrote equations to represent multiplicative relationships between two unknown quantities [25]. Others researchers claimed that the division of a unit into equal parts, are essential for developing rational number meaning [23, 29, 40].

For answer the above questions, this paper is presented in three research parts. The first research part aims to explore the difficulties faced by students in primary (pupils of the fifth and sixth grade of Primary School) and secondary education (students of first, second and third class of Middle School and first class of High School) over rational numbers, namely the notions of fraction sequences as representations on the number line and the notions of dividing the unit fraction into equal parts and improper fractions.

The second research part aims to identify the causes and reasons why students face these difficulties and what factors affect the presence or absence of said difficulties. Emphasis is given on the perceptions of prospective teachers over rational numbers, the structure and content of textbooks of mathematics in primary school, as well as on the teaching suggestions and approaches that several researchers have proposed on an international level.

The third and final research part is a presentation of proposals to reduce the causes of students' difficulties concerning the above notions. These proposals are presented in the form of instructional interventions using experiential activities and activities in online environments. In addition, the software Fraction Battles is presented, which was created by the research team to complement teaching interventions in order to reduce the students' struggles with fractions.

2. Literature Review

Fractions are one of the most difficult mathematical concepts for many students and they are among the essential concepts that students meet in school mathematics. Hence, many researchers have dealt with fractions on international level. In this point we present current literature on the subject of representations in fractions. In other words, which representations have emerged through these surveys as the most appropriate or inappropriate ones for the students to understand the notion of fractions [1].The findings of these researches concerning solely the number line, improper fractions and equal parts of the fractional unit, notions that this paper investigates.

More specifically, as far as the number line is concerned, Brousseau, Brousseau & Warfield [5] conducted a series of interventions in order to lead students day by day to invent, understand and become very good at all aspects of both basic mathematical structures, the rational and decimal numbers. The intervention included a total of 65 courses (15 cycles) which were held in the fourth grade of Michelet school. The courses were repeated in two parallel classes with different teachers in a period of over 15 years, which means that more than 750 students have taken part in them. In the third lesson of the fifth cycle a representation of fractions on the number line takes place, leading gradually to the representation of the number line in Figure 1, through a range of playful procedures and teaching methodology. According to this research, at the end of this activity, most students can quickly and undoubtedly put decimal fractions on the number line, and all students can analyze a decimal fraction in units, tenths, hundredths etc.

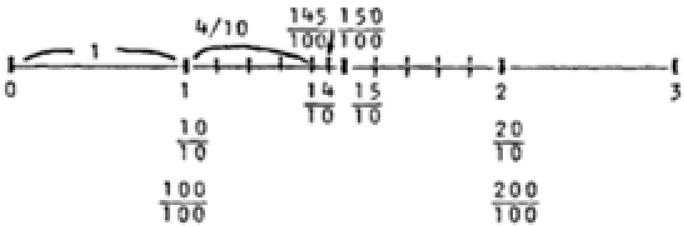

Figure 1: Finale form of number line according to Brousseau et al.

Another research proposal on the number line is that of Sedig & Sumner [36] who reported the importance of visual representations of mathematics and the use of digital tools that enable them. One of these tools is the use of zoom on the number line. Zooming raises or lowers the level of detail on the number line, allowing students to visualize dividing numbers into equal parts and thus facilitate the transition to the field of rational numbers.

Regarding the understanding of dividing a unit into equal parts, in their research, Olive & Vomvoridi [32] suggest teachers to avoid incorrect representations, where the division of the fractional unit in equal parts (see Figure 2a) is not frequently observed, which may lead students to believe that dividing the unit in equal parts in not necessary. Especially, Shahbari & Peled [38] proposed the use of an activity that involved fractions in a realistic situation with modeling characteristics (see Figure 2b) to promote understanding of dividing a unit into equal parts.The activity was successful not only in producing conceptual change in students' understanding of the changing reference in fractions, but was further generalized and transferred to percentages.

Figure 2: a) (*Left side*) representation to how to distribute 8 pizzas in 10 people. b) (*Right side*) the dimensions of the original model

As far as improper fractions are concerned, Hackenberg [16] used the JavaBars software on the approximation of the notion, noting the importance of the ability to create improper fractions for placing numbers on the number line, for the construction of fractional numbers that lead the way to developing a sense of consistency and continuity to the numbers.

3. Methodology

The researches followed a qualitative, quantitative approach. Moreover, a content analysis and case study were carried out. Thus a triangulation was formed, which was methodological, temporal, topical and theoretical in order to achieve stabilization of findings [8].

3.1 Participants

The research part of this paper lasted five years, took place in Greece and a total of 2052 participants took part in it (and tree research parts). The sample selection was a census, stratified and symptomatic according to the purposes and needs of the research.

3.2 Instrument

Regarding the data collection methods, questionnaires and tests written by researchers themselves were used, which reached their final form after pilot studies. Furthermore, some semi-structured interviews, video recordings, observation and literature study were employed.

3.3 Data analysis

In order to analyze the survey data, and in addition to the descriptive analysis, the Statistical Implicative Analysis by Gras, using the CHIC (Cohesive Hierarchical Implicative Classification) software [15] and Microsoft Excel program were used. The implication analysis of data was performed through similarity diagrams, in which the variables were associated with each other depending on the similarity or non-similarity they present. Variables in whose solution the subjects behave similarly are grouped together. The evaluation of software was based on the criteria set by Squires & McDougall [39].

3.4 Variables of research

The variables were defined as a combination of letters and one number. The letters indicate the initial of concept which is examined. For example, the variable NLi5a is composed of the initial proposal "Number Line" because the locating a number on a number line is examined and number 5a indicates the question of questionnaire.

4. The Researches

4.1 Research Part 1: The Perceptions of Students on Rational Numbers

The first research part consists of longitudinal surveys conducted from 2010 to 2014 and were designed not only to identify the difficulties of students in primary and secondary education concerning fractions, but also to explore the stability of the findings of these studies. It is an important research part for designing teaching interventions and educational software, as these findings were the basis and rules upon which the activities were planned.

This research involved a total of 992 students from Primary and Secondary Education, from public and private schools both in the capital and province. The implication analysis of data is illustrated by similarity diagrams in which the variables are associated with each other depending on the similarity or non-similarity they present (see Figures 3, 4 and 5). Based on the Figures 3, 4 and 5 we can say that students of both primary and secondary education seem not to have understood the importance of dividing the fractional unit in equal parts. Moreover, both educational levels present significant difficulties in placing fractions on the number line, difficulty which is intensified when the fractions are opposites.

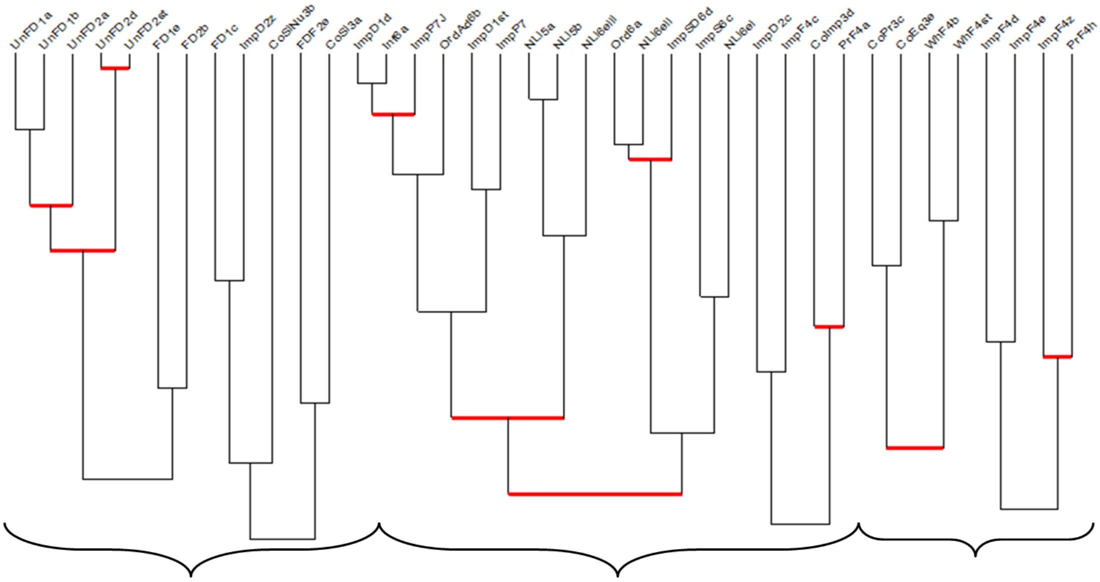

Figure 3: Similarity diagram: Results of primary studends' perceptions on fraction

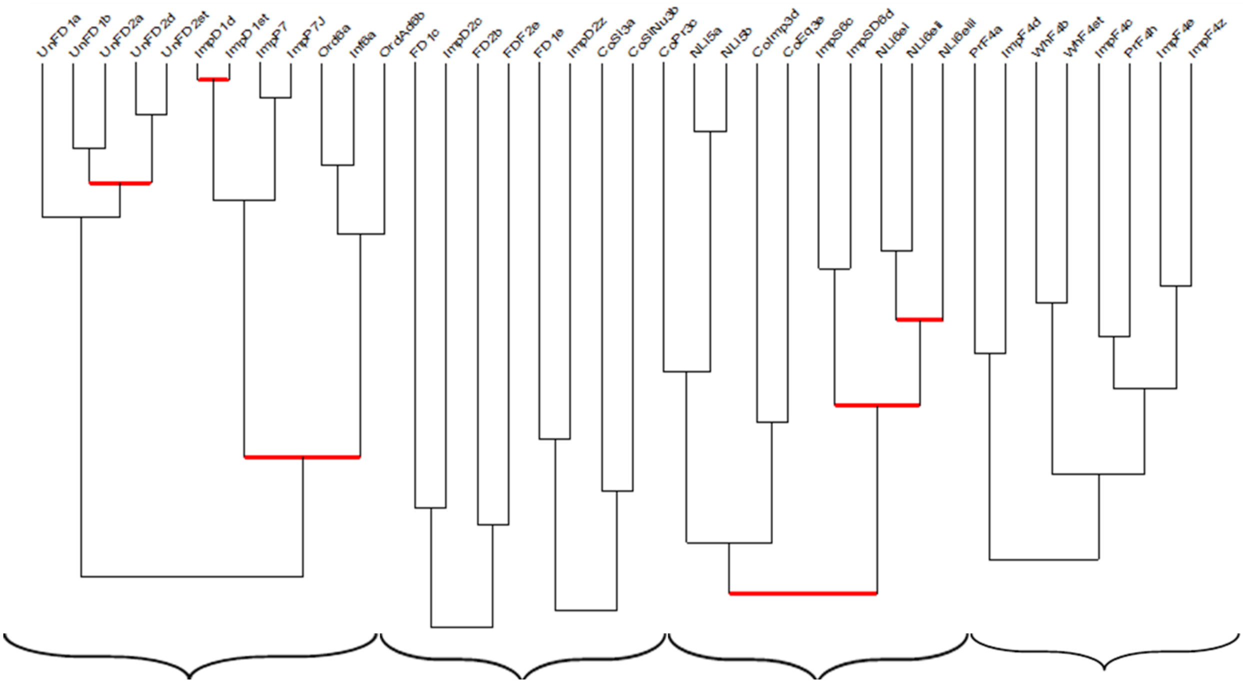

Figure 4: Similarity diagram: Results of middle studends' perceptions on fraction

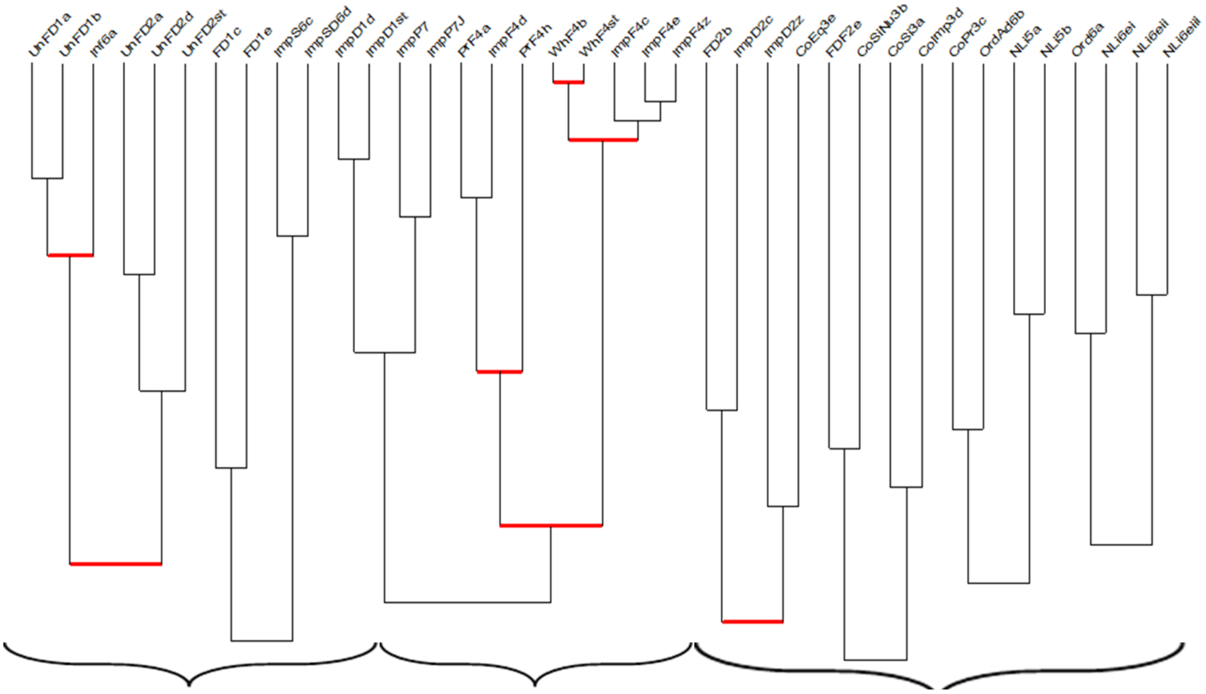

Figure 5: Similarity diagram: Results of high studends' perceptions on fraction

Other difficulties faced by students in both levels of education are finding a fraction between two consecutive fractions (e.g. 1/3 and 2/3) and understanding the density, sequence and infinity of fractional numbers [2, 44, 45]. Additionally, with respect to improper fractions, most students do not know their meaning and therefore cannot illustrate them. It is also common for children to believe that all fractions are <1 [35].

4.2 Research Part 2: The Perceptions of Prospective Teachers Concerning Rationals – Structure of Mathematics Textbooks

The second part of the research aims to identify the causes of these difficulties that students face. Several researchers have attributed to a different factor these difficulties which the students face in fractions. So, some researches point to the conclusion that the variety of representations in mathematics was associated with success in problem solving [14], especially, the variety of representations in mathematics the textbooks it is important to developing rational number meaning [20, 47]. In addition, some researches have stated that the perceptions of the teachers and the prospective teachers play a major role in these difficulties that students face, who must have deep knowledge base to support children's learning of mathematics [27, 24, 10, 43, 34, 42, 46]. The perceptions and beliefs of prospective teachers and the structure and content of Greek mathematics textbooks of primary school are therefore highlighted.

4.2.1 Perceptions of prospective teachers concerning rationals

The second research part consists of longitudinal surveys conducted from 2010 to 2013 which were designed to test the perceptions of prospective teachers on various notions of fractions, since it is they who will be asked to teach fractions to students and the success or failure of teaching will also depend on their own perceptions on fractions. This research involved a total of 900 prospective teachers from various parts of Greece. The implication analysis of data was performed through similarity diagram (see Figure 6), in which the variables were associated with each other depending on the similarity or non-similarity they present.

The results showed that prospective teachers face major difficulties both in finding a fraction between two fractions and finding fractions on the number line. Prospective teachers appear to show equally significant difficulties when it comes to improper fractions, as the works with these notions had the lowest success rates. On the notion of equally dividing the unit, it was observed that prospective teachers do not face any problem in finding the fraction representing a shape when the unit is divided into as many equal parts as the denominator says. They begin to experience difficulties, however, when the parts in which the unit is divided do not coincide with the denominator of the fraction. Consequently, it seems that prospective teachers have not fully grasped the notion of equal parts of the fractional unit (see Figure 6).

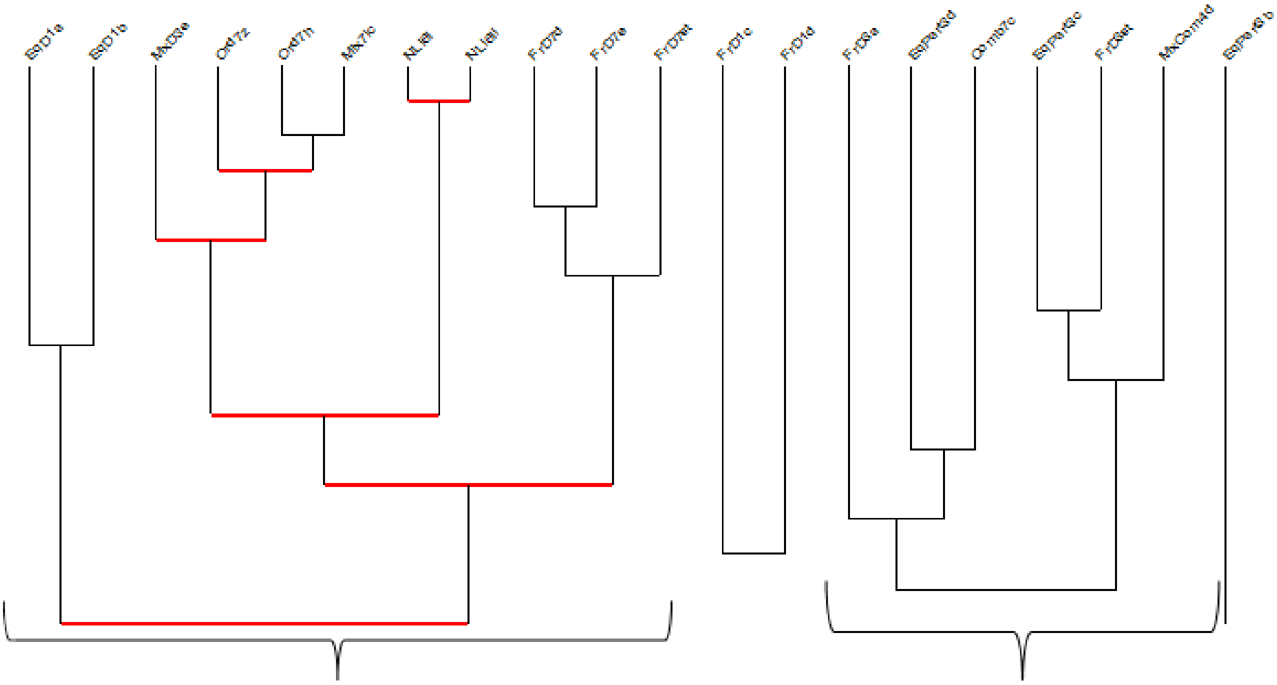

Figure 6: Similarity diagram: Results of perceptions of prospective teachers concerning rationals

4.2.2 Structure of mathematics textbooks

This research part also includes the content research conducted on Greek textbooks of mathematics of all six grades of primary school. The group of books studied includes student's books, activity books, and teacher's books- totally, 38 textbooks. The purpose of this research was to check the quality, quantity and appropriateness of representations on the notions of equal parts of the fractional unit, improper fractions and sequences of rational numbers on the geometric model of number line. At the same time, a commentary on the adequacy and type of representations and their correlation with the difficulties that students encounter takes place, based on surveys on these notions. Findings show a limited extent of the chapter of textbooks dealing with these notions, and limited implementation of multiple representations [4].

The significance of the findings from the survey of school textbooks can be understood through the findings of other international surveys. In particular, studies have shown that the more frequently a student comes in contact with a representational form, the more he/she familiarizes with it and the better he/she learns it [18, 21].

4.3 Research Part 3: Teaching Intervention – Creation of Software

The third part of the research aims to provide proposals to help to reduce the student difficulties. The way of teaching is a key factor influencing the future development of the notion of understanding in students' perceptions [28, 41, 37, 7, 33, 19], as well as and the representations which be used [11, 9]. Furthermore, as supporting other research, the fractional knowledge is effect by the way of teaching which the teachers adopt according to their type of cultural influence [6, 30, 31, 12].

Hence, taking under consideration the findings of the two previous research parts, teaching methods, both pilot and final, were designed and implemented for the notions of number line, improper fractions and division of the fractional unit in equal parts. The sample consisted of 160 participants. The teaching interventions were implementing in students of fifth and sixth grade primary and prospective teachers. Four classes were designated-3 classes of fifth and sixth grade primary and one class of prospective teachers. The teaching interventions were made at different times for each class and they lasts for two weeks each one.

The research tools were questionnaires/tests, semi-structured interviews, video recordings, teaching interventions and observation, and analysis was performed through the statistical package CHIC.

4.3.1 Teachings description

The final teachings were divided into seven phases.

(a) The 1th- 3th phase

The first phase included the completion of written tests before teaching. The second phase included semi-structured interviews of the participating students. The third phase included teaching that was designed to expose students to as many multiple representations as possible that depicted diagrams of various shapes, using counter-examples as well, and were related to the notions of equal parts and improper fractions. The presentation of the representations was made with Microsoft Power Point and Microsoft Word software and included 55 multiple representations in total (see Figure 7).

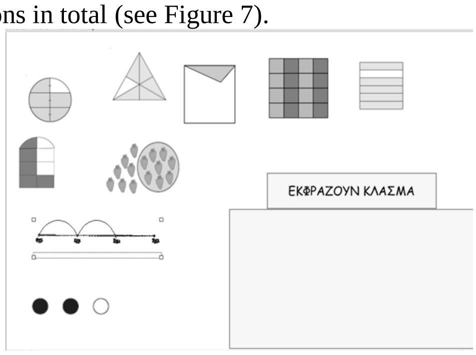

Figure 7: Multiple representations during 3th phase of lectures

(b) The 4th phase

The fourth phase was comprised by teaching involving placing the fractions on the number line using experiential representation (see Figure 8a), and the representation on an electronic environment using the software ConceptualMath:Place Fractions on a Number Line (Figure 8b).

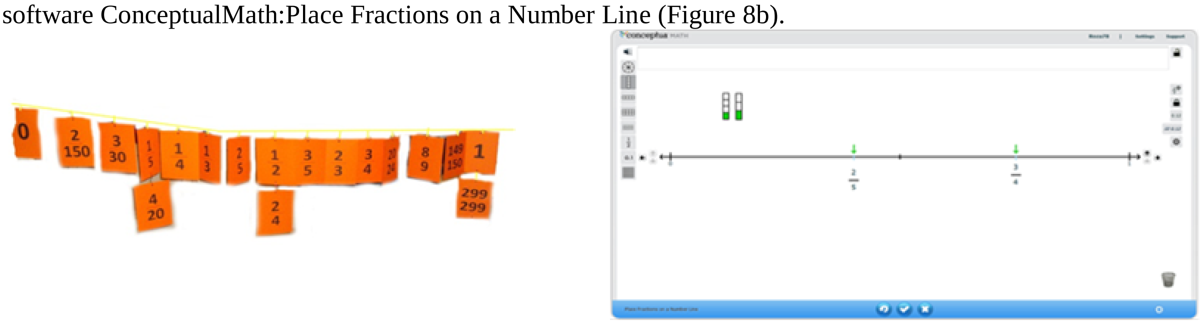

Figure 8: a) *(Left side)* activities for number line using experiential representation. b) (Right side) activities for number line using the software ConceptualMath

A rope was placed on the board representing the number line (see Figure 8a) on which the points 0, 1 and 2 were set. Several tabs were given to students showing different fractions, either improper, proper, whole etc. and each student came to the board in order to place his/her tab correctly on the number line, with reference to existing points, at the beginning only 0, 1 and 2, and then to the students' tabs that had been placed on the number line as well. Before the students placed their tab in the number line, they had to report to their classmates, externalizing their thoughts, the reason why they intended to place the tab at this point of the number line.

Some of the students' arguments were:

For 3/4: The numerator is smaller than the denominator, so my fraction is smaller than the unit, so I will place it before 1.

For 299/299: The numerator is equal to the denominator, so it is equal to the unit, so I'll place it under 1.

For 2/150: 2 is very far from 150 which is my unit, so my fraction is close to 0.

For 1/4 (students had already placed 1/5 and 1/3 on the number line): My fraction has the same numerator with 1/5 and 1/3 so it goes between them, since in fractions with same numerator, the one with the smallest denominator is the greatest.

For 149/150: My fraction is very close to the unit so I'll put it next to 1.

For 8/9: My fraction is very close to unit, it only needs one piece to be 9/9 but this piece is larger than the piece of 149/150 which also needs one piece to be 150/150 so I'll put it next to 1, before 149/150.

For 5/4: In my fraction the numerator is greater than the denominator, so my fraction is greater than the unit so I'll put it after 1.

In any explanation given by the students for their choice, a debate takes place first between the group and then between all groups for whether the choice of each student is correct or not. The confirmation, questions and errors are corrected by the students themselves, using the software Conceptual Math → Place Fractions on a Number Line (see Figure 8b).

(c) The 5th phase

The 5th phase included the implementation of educational software-Fraction Battles- (see Figure 9a and b) created by the research team. This software is related to the notions of equal parts of the unit, improper fractions and sequences of rationals in the geometric model of the number line. The aim of the software is to familiarize students with rational numbers and help them minimize the difficulties they face when it comes to fractions by using multiple representations on which the added value of the software is also based, through a wide range of activities of a dynamic multimedia environment. For its design, all findings from longitudinal surveys from 2010 to 2015 described in the previous sections were taken into account.

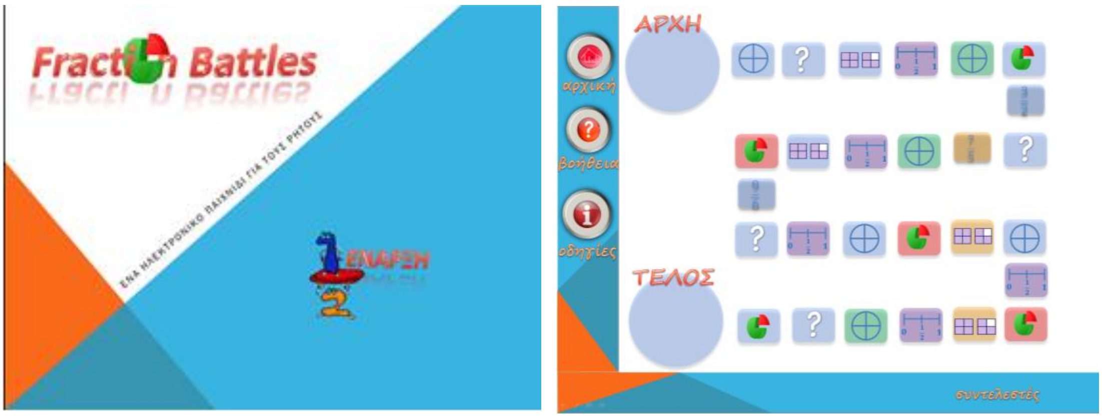

Figure 9: a) (*Left side*) home page of Fraction Battles. b) (*Right side*) digital dashboard of software Fraction Battles with 27 activities

The route includes 27 points/activities (see Figure 10a, b and c). Each activity is designed to refute some of the difficulties students have in rationals, as highlighted by previous research. Thus, each activity refers to a specific notion, has a specific aim and must fill in specific gaps. In addition, activities are graded by difficulty.

(d) The 6th and 7ht phase

The 6th phase included the completion of tests after teaching in order to check the degree of teaching "intrusiveness" and in the 7th phase semi-structured interviews of participating students were performed after the teaching interventions.

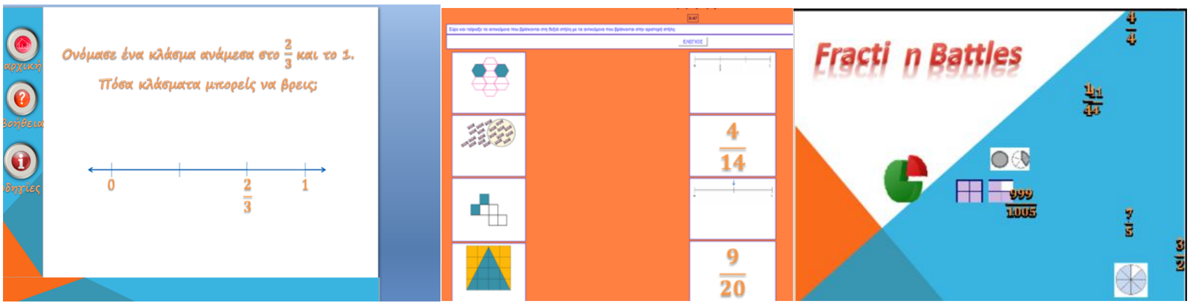

Figure 10: a) (*Left side*) activity for the sequence and density of rational numbers. b) *(Middle side)* activity for transfer from one form of representation to another, regarding the division of fractional unit in equal parts. c) (*Right side*) activity for improper fractions and their management from one form of representation to another

4.3.2 Results of research part 3

After the teaching interventions, an improvement in students' performance was observed from data analysis (see Figure 11, 12). In particular, in finding fractions on the number line, the success rate was initially 24% and 14% (NLi6i NLi6ii, Figure 11) while in the post-activities rates, it rose to 64% and 61% respectively (Figure 12).

On the notion of unit division into equal parts, we also observed increased success rates, since in exercises concerning recognizing fraction in shapes that were not divided into equal parts, rates from 0% in pre-activities, rose to 52% in post-activities.

Finally, we observed increased success rates in the notion of improper fractions as well, as they rose from 18% to 64% (MxD3e) in recognizing improper fraction from a diagram. The increase in success rates following teachings in the exercise asking students to schematically represent the fraction 10/4 (Mix7ic) was also impressive. Success rates increased from 15% in pre-activities to 55% in post-activities.

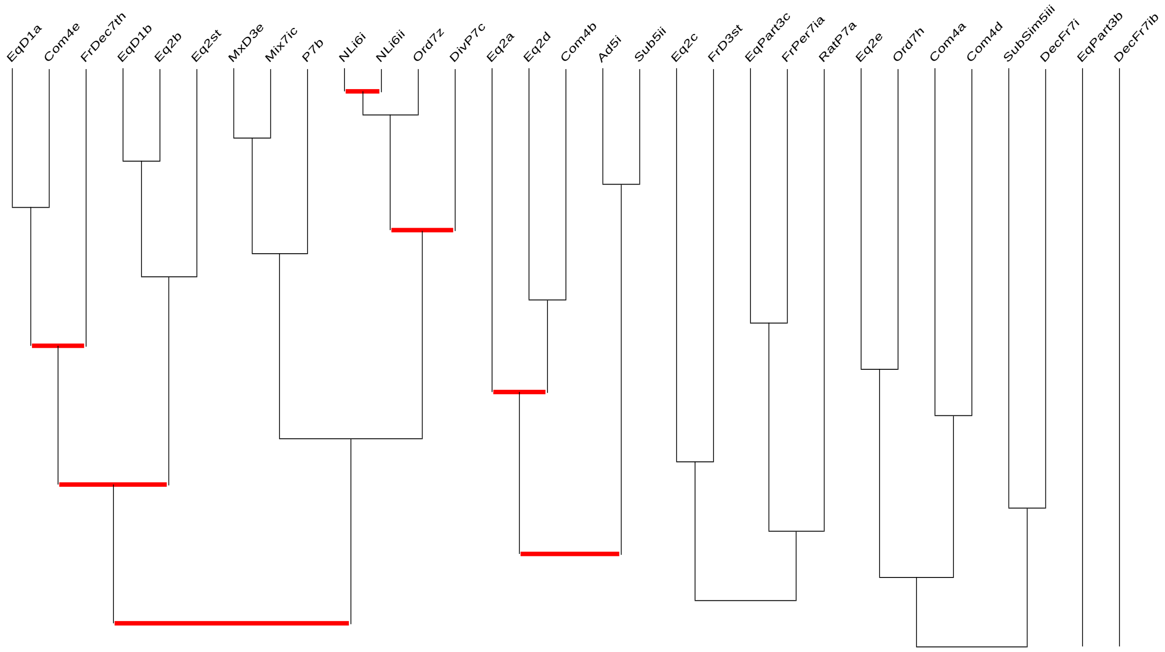

Figure 11: Similarity diagram: Results of pre-test

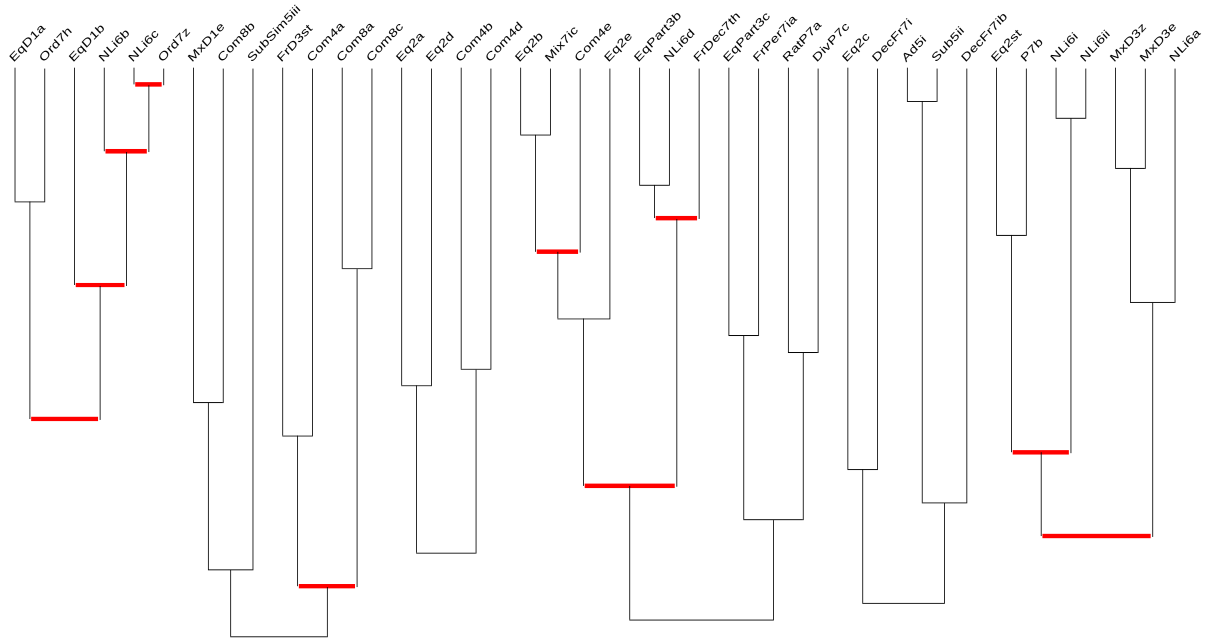

Figure 12: Similarity diagram: Results of post-test

5. Discussion

Rational numbers are an important part of our students' mathematical literacy, as their understanding further contributes to understanding other mathematical notions [22, 17, 3, 25, 26]. For this reason, many researchers move in this area by investigating the students' difficulties over rationals. Several of these studies share a common component: the idea that the way of teaching is a key factor influencing the future development of the notion of understanding in students' perceptions. Other researchers argue that the more frequently a student comes in contact with a representation form, the more familiar he/she becomes with it and the better he/she learns it [18, 21].

We see, that is, that several researchers have attributed these difficulties that students face with rational numbers to different factors. Despite the research efforts, the constant changes in books, the suggestions etc. students still face the same difficulties and the student population presents the same discouraging picture concerning rational numbers in all levels of education.

Thus, we provide instructional practices how the use of multiple representations helps students to reduce their difficulties on fractions and its positive effect on fraction understanding. The analysis of the collected data both similarity statistical method showed that multiple representations and the appropriate teaching helped enough students to reduce their difficulties about classification of fractions as a representation on the number line, as well as the concepts of the unit's division in equal parts and the concept of the improper fractions.

In conclusion, our research team, believing that these difficulties of students are due to a combination of factors including, among other things, the findings of the aforementioned studies, attempts, through a longitudinal research that combines all possible causes mentioned by several researchers as well as new ones that raised, to highlight these factors with the ultimate aim of formulating and suggesting substantive solutions to address these students' difficulties.

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